One-way heat flow formulation with Planck absorption and re-emission

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One-way heat flow formulation with Planck absorption and re-emission

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updated September 2014, updated MATLAB listing

Introduction

In recent papers [1, 2, 3] the author has discussed the one-way versus the two-way formulation for heat flow by radiation.

In [1] is presented a model of the semi-transparent atmosphere consisting of a stack of gauzes, representing the IR-active trace gases (molecules with three or more atoms). Results have been validated by comparing them with published results of K&T type of diagrams based on the two-way formulation. The one-way formulation does not show the huge absorption in the atmosphere nor the so-called back-radiation therein.

In [2] a finite element implementation is presented for the one-way heat transport in a stack of gauzes, with the same results as [1] concerning OLR and sensitivity due to doubling CO2.

In [3], again by means of a finite element technique, the heat transport by conduction through two slabs with a finite thickness separated by a vacuum with a radiation heat transfer shows once more that the back-radiation does not exist.

In [1] and [2] the heat transport by radiation is studied for an atmosphere with given temperature distribution defined by the environmental lapse rate. The absorption and re-emission of heat by IR-active molecules is approached by a stack of grids of “black” wires.
In this paper, instead of a stack of “black” grids, the absorption and emission in the atmosphere is assumed to be given by the Planck function as, for example, described in the Science of Doom (SoD) blog [4, 5], however, without the two-way heat flow formulation.

In appendices 2 and 3 the two-way formulation of Schwarzschild equation, as presented in SoD [4,5] is analysed in detail. It reveals that the Schwarzschild approach in fact is a scheme identical to the one-way formulation of heat exchange between pairs of emitters and absorbers.
The same type of equations as in the stack model results: the absorption coefficients of the gauzes in the stack model can be replaced by Planck functions. However, in those equations, appear view factors which in the Schwarzschild are calculated in a different way as compared to the stack model.
These view factors have an impact on the results and indicate that the stack model is to be preferred.

In appendix 4 slight differences are discussed concerning so-called view factors.
In appendix 5 the listing of a MATLAB program is given, with line by line green comments.

Schwarzschild equation

With reference to the SoD blog [4, 5] we repeat the so-called Schwarzschild equation, applied for up-ward and for down-ward fluxes, in the two-way heat flow formulation in a semi-transparent atmosphere.

In plain words

It is supposed that the radiation can be split into up-ward components U and a down-ward ones D . Radiation strength depends on the wave length and we can consider U and D as components over a wave length interval at a certain wavelength.

The dimension of the different U's and D's is W/m^2, we call them fluxes.

The up-ward flux starts at the surface of the planet with the Prevost type of flux corresponding to the surface temperature T_s and emissivity according to Stefan-Boltzmann: $\gamma(\varepsilon\sigma T_s$^4) where γ represents the fraction of the Stefan-Boltzmann expression related to the wavelength interval and wavelength of the U components.

On the one hand, the up-ward flux decreases proportionally to the flux itself due to absorption by the presence of IR-active molecules along its path upward.

On the other hand it increases by the emission of the same IR-active trace gases proportional to the Planck function B (T(z)) for the wavelength interval and wavelength of U and to the number of IR-active molecules, assuming that they are emitting as if the molecules were radiating towards outer space. The process is described by a Schwarzschild equation for an upward flux component U.

For the downward components D , taken positive in the negative z-direction, a similar process can be described by a Schwarzschild equation. The downward component analysis starts with a LW input at the top of the atmosphere, TOA, usually taken as zero.

The downward component D, on the one hand starts to increase along its path s= Htoa-z from z = Htoa down to the surface z=0, due to downward emission of IR-active molecules at a temperature T(s) and Planck function B (T(s)).

On the other hand, it decreases due to absorption of a fraction of the downward component.

In more strict mathematical terms.

The up-ward components are supposed to be described by the following linear differential equation:

$$dU = - N\alpha(U- B)dz \tag{1}$$

dU	variation of upward component with height	[Watt/m^2]
N	number IR-active molecules per cubic meter	[1/m^3]
α	cross-section for absorption	[m^2]
B	Planck function component for emission	[Watt/m^2]
dz	variation of vertical co-ordinate z	[m]

A similar equation can be written for the down-ward components D, which are taken positive in down-ward direction:

$$dD = N\alpha(D- B)dz \tag{2}$$

In (1) and (2) the parameters U , D , and B(T) are for a one centimetre wavelength interval at a specific wavelength.
The total upward intensity, the total downward intensity and the total Planck function , all three function of the height z, become the sum of the components.

$$U_{tot} = \Sigma U \quad (3) \qquad D_{tot} = \Sigma D \quad (4) \qquad B_{tot} = \Sigma B(T) \quad (5)$$

For the surface of the planet with temperature T_s and emissivity ε, it is the so-called Prevost term:

$$U_{tot}(z=0) = \varepsilon\sigma T_s{}^{\wedge}4 \qquad (6)$$

In order to study in a transparent way the proposals of Schwarzschild we replace the parameters Nα by β being absorption per unit of length.
The Schwarzschild procedure, to split up artificially the radiation in up-ward and down-ward components at a certain wave length, can then be studied by looking to the solutions of the system of the linear differential equations with β a function of z:

$$dU/dz = -\beta(U - B) \qquad (7)$$

$$dD/dz = \beta(D - B) \qquad (8)$$

In appendix 3 the classical original analytical Schwarzschild solution is given by introducing a coordinate transformation based on so-called optical thickness, as well as two other techniques to deal with the equations numerically.

Semi-transparent grids to model the atmosphere.

Suppose we have an atmosphere with a number of axial stations including the surface node = 1 and the outer space node = nods, with absorption coefficients f_i and temperatures T_i .
Between the nodes we can identify nods*(nods-1)/2 pairs which are communicating with each other.

Heat exchange between the grids by means of finite elements

At each node we have an absorption coefficient f_i and a temperature T_i .
The heat exchange by radiation between two surfaces is given by the generalized Stefan-Boltzmann law [3] based on the work of Christiansen, 1883, [6]:

for $T_i > T_j$ $$q(i\rightarrow j) = \varepsilon_{ij}\sigma(T_i{}^{\wedge}4 - T_j{}^{\wedge}4) \qquad (9)$$

$$1/\varepsilon_{ij} = 1/\varepsilon_i + 1/\varepsilon_j - 1$$

The latter relation can also be found in Wikipedia:
http://en.wikipedia.org/wiki/Emissivity#Emissivity_between_two_walls

In plain words, surfaces exchange with each other information about their temperature **and** about their surface conditions, on the basis of which heat is exchanged, emitted by the surface with the higher

temperature and absorbed by the surface with the lower temperature.
In practical terms, in the radiation process we have to identify pairs of emitters and absorbers (i, j) with temperatures T_i and T_j, not necessarily adjacent to each other.
In a semi-transparent atmosphere, between two communicating surfaces defined by nodes i and j there might be other semi-transparent surfaces k in between, communicating with i and j.
In analogy with (9) and introducing nodal parameters $\theta_i = \sigma Ti^4$ we can write for the nods*(nods-1)/2 pairs the heat exchange between nodes i and j<nods [2]:

$$\theta_i = \sigma Ti^4 \quad , \qquad q(i \rightarrow j) = fe_{i,j} * (\theta_i - \theta_j) \ , \qquad fe_{i,j} = f_i * viewfactor(i,j) * f_j \tag{10}$$

$$\text{with} \qquad viewfactor(i,j) = 1 - \sum_{k=i+1}^{k=j-1} f_k$$

Alternative view factors are presented in appendix 4.
Each pair represents a finite element with two nodes and characteristic equation with $fe = fe_{1,2}$:

$$\begin{vmatrix} fe & -fe \\ -fe & fe \end{vmatrix} \begin{vmatrix} \theta_1 \\ \theta_2 \end{vmatrix} = \begin{vmatrix} q_1 \\ q_2 \end{vmatrix} \tag{11}$$

The nods*(nods-1)/2 overlaying finite elements can be assembled into a system matrix **K**:

$$\mathbf{K}*\boldsymbol{\theta} = \mathbf{rhs} \qquad \text{with the solution} \qquad \boldsymbol{\theta} = inv(\mathbf{K})*\mathbf{rhs} \tag{12}$$

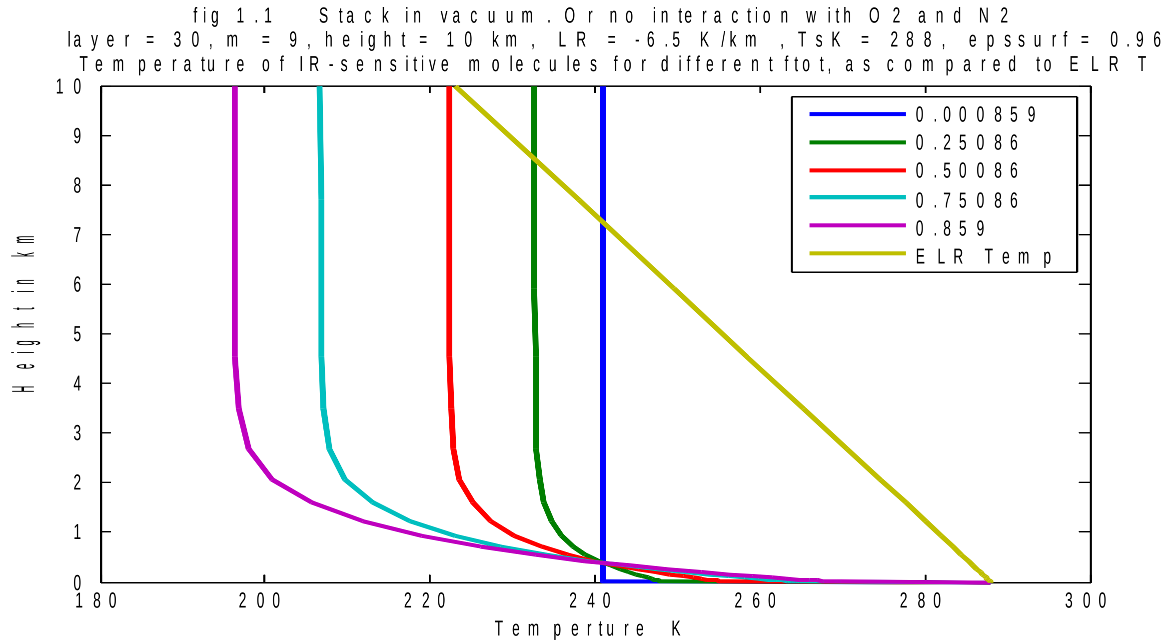
K : system matrix of order nods x nods,
inv(**K**) : inverse of **K** taking into account boundary conditions
$\boldsymbol{\theta}$: vector of unknowns $\boldsymbol{\theta_i}$ in W/m^2, of order nods
rhs : right hand side vector of fluxes **q** in W/m^2, into the system, of order nods

In case there would be no heat transfer from the bulk of the atmosphere to the IR-active trace gases by mechanisms other than LW radiation the right hand side **rhs = 0.**
The unknown $\boldsymbol{\theta}$ are then defined by (12) with the boundary conditions T=288 K at the surface node=1 and at outer space node = nods, T=0 K.

In figure 1 the results are depicted for such an hypothetical situation.
The resulting temperature distribution is given for different values of ftot, which represents the total amount of absorbers in a column of air.
The value ftot is also called the optical thickness of the atmosphere.

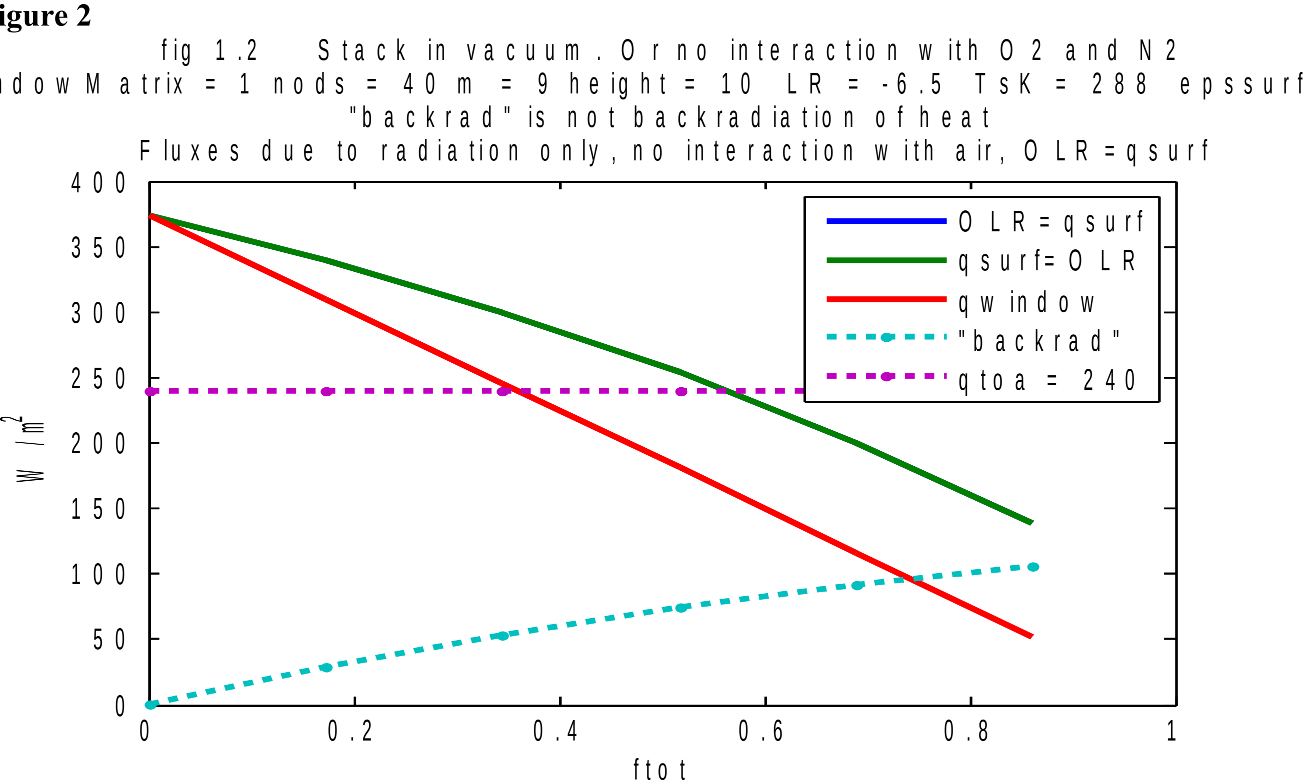
As reference is included the temperature due to the environmental lapse rate, ELR.
The results of the hypothetical atmosphere as presented in figure 1 are of interest!
It is noted that in case there is no interaction with the other molecules of the bulk of the atmosphere with a temperature defined by ELR, the IR-active trace gases are much colder than the atmosphere.
For the annual and global mean atmosphere , known as the K&T atmosphere, the total absorbers are represented by ftot = 0.86 [1]. See also appendix 4 on the sensitivity of this analysis.

Figure 1



The conclusion is that IR-active trace gases do not heat up the atmosphere!
It is the other way around, the 99% bulk of the atmosphere keeps, by molecular collision also called conduction, the IR-active trace gases warm!
Figure 2 gives the OLR for this hypothetical case. For ftot=0, the OLR = qsurf flux is equal to the Prevost flux = $\varepsilon\sigma T_s$^4 , because without IR-active trace gases the surface is indeed looking without any absorption to outer space at zero K.

Figure 2



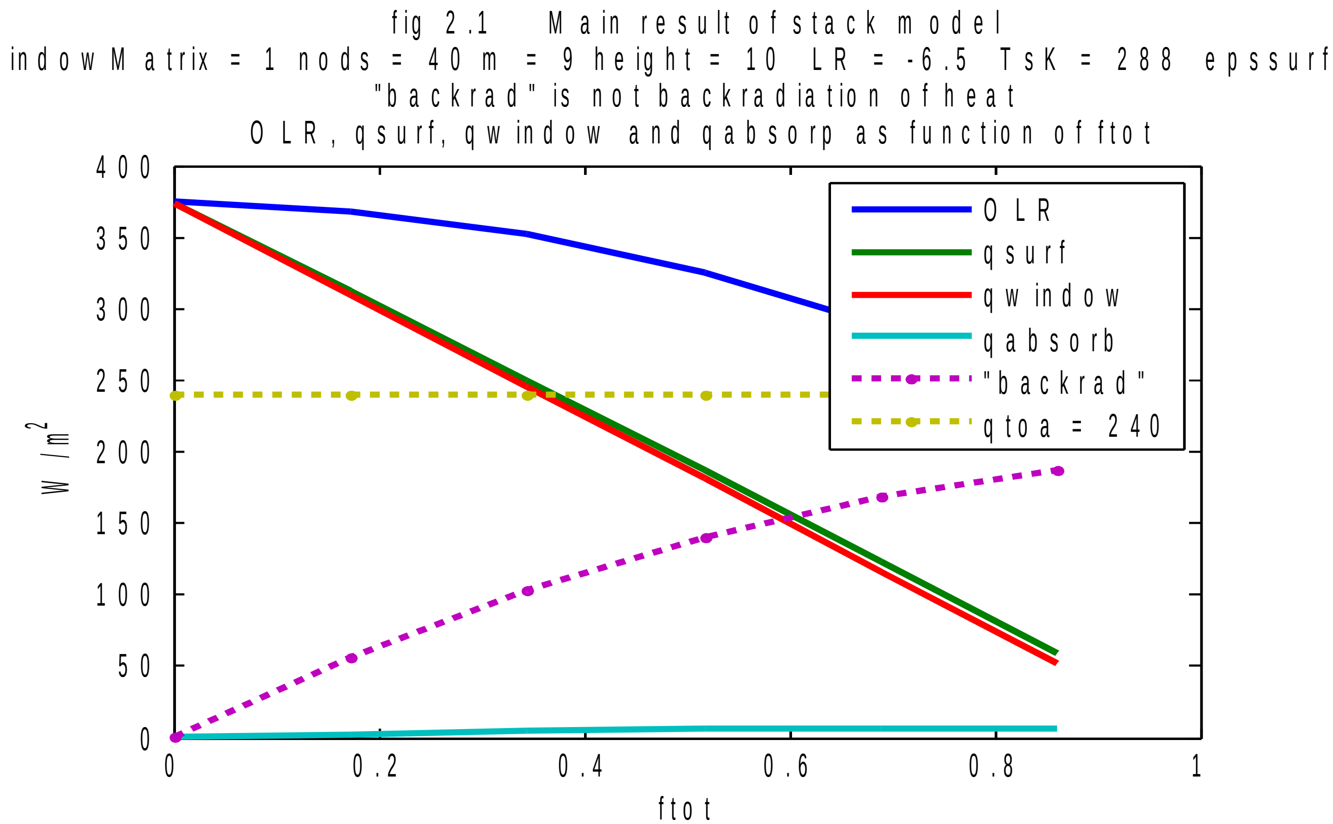
The flux through the atmospheric window is given by qwindow = (1-ftot) $\varepsilon\sigma T_s$^4.
In this hypothetical case , without heat transfer between IR-active trace gases and the bulk of the remaining 99% of the atmosphere, OLR=qsurf.
In appendix 5 a listing is presented of a MATLAB program by which figures 1 and 2 are generated. The reader can make his own runs with different boundary conditions.

An alternative and probably more important use of the matrix relation (12) is the reverse solution technique. With a given temperature (and thus θ) distribution:

$$\mathbf{q} = \mathbf{K}^*\theta \qquad (13)$$

q **:** vector of necessary heat fluxes by mechanisms other than LW radiation (convection, wind, SW-absorption, aerosols etc.) to obtain a measured temperature distribution defined by θ **.**

The results in figure 3 represent the standard output for the stack model as presented in [1,2] and generated by the MATLAB program of which the listing is given in appendix 5.
In that program, option 2 represents the K&T global and mean atmosphere. In those publications the OLR is said to be 240 W/m^2 and that resulted into a value of ftot = 0.86. [1]
Figure 3



The dotted curve “backrad” does not represent a back radiation of heat.
In the finite element program with total number of nodes = nods there are (nods-1) elements with lower node=1. The qsurf curve represents the heat flow radiated by the surface and is represented by the sum of the heat fluxes in elements with nodes (1,j):

$$\text{qsurf} = \sum_{j=2}^{j=nods} f(1)*\text{viewfactor}(1, j)*f(j)*(\theta(1) - \theta(j)) \tag{14}$$

The hypothetical back-radiation is defined by the sum of the highlighted terms:

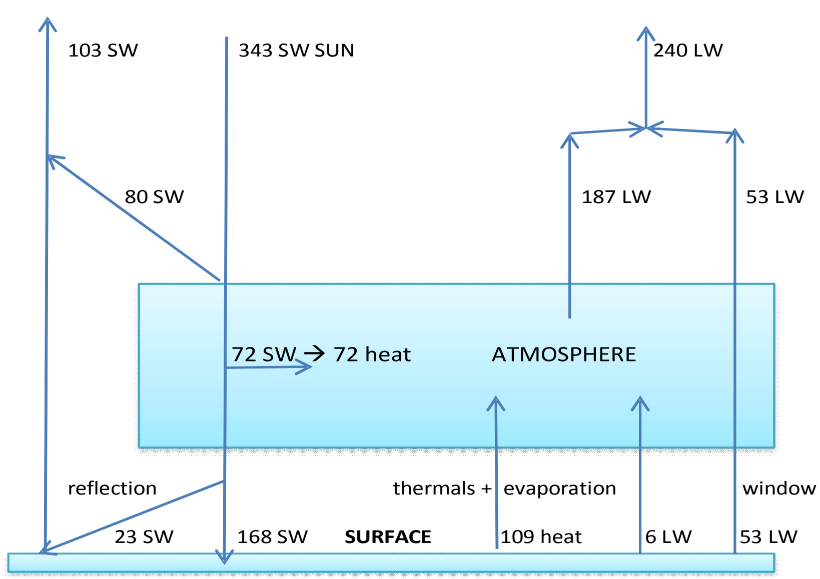
$$\text{“backrad”} = \sum_{j=2}^{j=nods} f(1)*\text{viewfactor}(1, j)*f(j)*\ \theta(j) \tag{15}$$

This algebraic expression is not a flow of heat, it is just a term in the relation for the surface heat flux (14). It is noted that the surface flux (14) is not of the Prevost type, it depends on all nodes and and all temperatures!

We see that the absorption in the atmosphere represented by the difference between qsurf and qwindow is small. The heat transfer mechanism is mainly due to other mechanisms than LW absorption.
That difference being the heat which is deposited by other mechanisms than LW radiation is represented by the difference between OLR and qsurf.

The resulting heat budget based on the option 2 values is given in figure 4, with OLR =240 ,
a surface flux of 59 of which 53 through the window, only 6 absorbed by the atmosphere.
The heat evacuation from the surface to higher regions by convection is 109. [2]

Figure 4
Global and annual mean heat budget in W/m^2



In appendix 4 an alternative analysis is given for this global and annual mean heat budget, evaluated for different viewfactors.

Heat absorption and re-emission based on Planck

The Planck function can in each node (i=1:nods) be defined for wavelength intervals at different wavelengths (k=1:ktot) and at a temperature Ti. In total ktot*nods variables B(i,k).
At each node i with and wave length k we have an height h(i), a number N(i,k) of IR-active molecules at wavelength k, and an emissivity α(k) giving an expression for an element flux with nodes i and j defined by three algebraic expressions, similar to(10)

$$f(i,k) = h(i)*N(i,k)*\alpha(k) \tag{16}$$

$$f(j,k) = h(j)*N(j,k)*\alpha(k) \tag{17}$$

$$fe(i,j,k) = f(i,k)*viewfactor(f(i,k),f(j,k))*f(j,k) \tag{18}$$

Element matrix for wave lenght k similar to (11)

$$\begin{vmatrix} fe(i,j,k) & -fe(i,j,k) \\ -fe(i,j,k) & fe(i,j,k) \end{vmatrix} \begin{vmatrix} B(i,k) \\ B(j,k) \end{vmatrix} = \begin{vmatrix} q(i,k) \\ q(j,k) \end{vmatrix} \tag{19}$$

The elements of wave length k can be assembled into a matrix $\mathbf{K_k}$ of dimension nods x nods similar to equation (12) .

$$\mathbf{K}_k*\mathbf{B}_k = \mathbf{q}_k \tag{20}$$

The similar relation of (13) becomes:

$$\mathbf{q} = \sum_{k=1}^{k=ktot} \mathbf{q}_k = \sum_{k=1}^{k=ktot}(\mathbf{K}_k*\mathbf{B}_k) \tag{21}$$

For known $\mathbf{K_k} *\mathbf{B_k}$ we find the vector $\mathbf{q}$ representing the mismatch of the LW radiation: more LW out of the atmosphere than into the atmosphere.
It is the heat deposit due to other mechanims than LW : convection of sensible and latent heat, absorption by aerosols etc.

This model is a one-way heat flow formulation.
It has already been validated in [1] as a mono-chromatic model by comparison with the results of so-called K&T diagrams, showing that the huge absorption in the atmosphere does not exist, nor the back-radiation of heat.
The foregoing discussion shows that indeed the stack model is qualitatively similar to the Planck approach. But the Planck approach enables to quantify the distributions of the different IR-active trace gases like water vapour , CO_2 etc. , by defining absorption coefficients f_i_H_2O , f_i_CO_2 etc.
The viewfactor matrix in (18) can be taken as "ones(nods)" except for the last column as is discussed in appendix 4. It remains a subject of further research.

Conclusion

IPCC authors use the Schwarzschild equation with the so-called two-way heat flow formulation using auxiliary fluxes U and D as is described in the SoD blog [4] and [5].
The auxiliary downward flux D is called back-radiation.
In appendix 2 and 3 the numerical procedures related to the two-way Schwarzschild formulation are discussed in more detail and compared to the one-way FEM formulation.
The analyses of appendix 2 and 3 show that the proposal of Schwarzschild does not give the correct results.

The exchange of heat between two absorbing/emitting layers is proportional to the difference of the corresponding Planck functions: f(i)*viewfactor(i , j) *f(j) * (B (Ti) - B (Tj))
NB When $T_i = T_j$ the exchange of heat is zero, as it should be according to the Second Law of Thermodynamics.

The numerical procedure has already been applied in [1] and in [2], not with Planck functions B but by means of a stack of semi-transparant grids of "black" wires with $\theta = \sigma T^4$.
In appendix 5 the listing of a MATLAB program is presented.

Acknowledgement

The author wants to thank in particular Claes Johnson who inspired him to write this paper.
The author interpreted his ideas by writing Stefan-Boltzmann always for a pair of surfaces: it opens the concept of standing waves.
The help of Hans Schreuder to edit this paper, to organize a peer review and host it on his site is acknowledged.

The author is indebted to the SoD blog owner who in an earlier version of this paper drew the attention to an unfortunate misinterpretation of the Beer/Lambert relation.
However, the Schwarzschild/Beer-Lambert proposal as presented in SoD blog (equations and code [4,5]) which are evaluated in the present paper, does not give coherent results, not even with the modification of the boundary conditions.
Unfortunately, SoD does not give results in the blog, only equations and a computer listing.

References

[1] http://www.tech-know-group.com/papers/IR-absorption_updated.pdf

[2] http://principia-scientific.org/publications/PROM/PROM_REYNEN_Finite_Element.pdf

[3] http://www.tech-know-group.com/papers/Prevost_no_back-radiation.pdf

[4] http://scienceofdoom.com/2011/02/07/understanding-atmospheric-radiation-and-the-%e2%80%9cgreenhouse%e2%80%9d-effect-%e2%80%93-part-six-the-equations/

[5] http://scienceofdoom.com/2013/01/10/visualizing-atmospheric-radiation-part-five-the-code/

[6] C Christiansen, Annalen der Physik und Chemie, Leipzig, 1883, page 2

Appendix 1

What do pyrgeometers measure?

As a reaction to earlier version of this paper it has been brought up that the conclusion that back-radiation of heat does not exist, can't be correct because “back-radiation has been measured”.

To explain how pyrometers work it is sufficient to look to the generalized Stefan-Boltzmann equation as given in (9) of the main text:

for $T_1 > T_2$ $$q(1 \rightarrow 2) = \varepsilon_{12}\sigma(T_1\text{^}4 - T_2\text{^}4) \qquad \text{(A1.1)}$$

$$1/\varepsilon_{12} = 1/\varepsilon_1 + 1/\varepsilon_2 - 1$$

In plain words: two remote surfaces exchange information with each other concerning their temperatures and their emissivities, on the basis of which a heat flow is exchanged from the warmer surface to the colder one.

Pyrgeometers use above equation.
Surface 1 is the sensor surface of the pyrgeometer with:

- known temperature T_1,
- known ε_1
- known σ ,
- known electrical input q.

Unknowns are the data of a remote surface to be measured: ε_2 and T_2.

The manufacturers are clever enough to include e.g. a value for ε_2 in the chip to measure the remote T2, or they make two measurements with two different sets (T_1, ε_1, q).

The gadgets give the possibility to display the remote temperature T_2 or the back-radiation which seems to correspond to $\sigma T_2\text{^}4$. That means that it is supposed that $\varepsilon_{12} = 1$ or $\varepsilon_1 = \varepsilon_2 = 1$.
It is not a back-radiation of heat.

Appendix 2

Two layer comparison of stack model and Schwarzschild procedure

Both for the Schwarzschild procedure and for the stack model, for a simple two-layer model the various equations can be followed up line by line, and compared to each other.
In the model of figure A2.1, node 1 represents the surface of the planet, node 4 outer space, node 2 and 3 the atmosphere.
We introduce 4 temperatures Ti and corresponding parameters $\theta i = \sigma Ti\text{^}4$ with dimension W/m^2.

The parameters θi are a measure for temperature, however they are also used as flux.
It follows from the context. The absorption factors are fi.

Two-way Schwarzschild formulation for two layers

In figure A2.1 are included the calculation of the **upward fluxes U_i** and the **downward fluxes D_i** according to the two-way Schwarzschild formulation.

Figure A2.1

-----------θ_4 = 0, f_4=1 **hypothetical downward flux** **hypothetical upward flux**

$D_3 = 0$ **$U_3 = (1-f_3)U_2 + f_3\theta_3$**

-----------θ_3 f_3 **$= f_1(1-f_2)(1-f_3)\theta_1 + f_2(1-f_3)\theta_2 + f_3\theta_3$**

$D_2 = (1-f_3)D_3 + f_3\theta_3$ **$U_2 = (1-f_2)U_1 + f_2\theta_2$**

------------θ_2 f_2 **$= f_3\theta_3$** **$= f_1(1-f_2)\theta_1 + f_2\theta_2$**

$D_1 = (1-f_2)D_2 + f_2\theta_2$ **$U_1 = f_1\theta_1$ hypothetical surface flux**

------------θ_1 f_1 **$= f_2\theta_2 + f_3(1-f_2)\ \theta_3$** **of Prevost type**

back-radiation

The flux to outer space and the back-radiation become according to Schwarzschild:

$$\text{OLR} = U_3 = f_1(1-f_2)(1-f_3)\theta_1 + f_2(1-f_3)\theta_2 + f_3\theta_3 \qquad \text{(A2.1)}$$

$$\text{back-radiation} = D_1 = f_2\theta_2 + f_3(1-f_2)\ \theta_3 \qquad \text{(A2.2)}$$

One-way Finite Element formulation for two layers

The one-way formulation by means of finite elements is given in [2].
In the main text of this paper an overview is given in equation (12).
For the 4 nodes we have 6 communicating elements with fluxes given by (12):

element (1, 2): flux = $f_1 f_2(\theta_1 - \theta_2)$

element (1, 3): flux = $f_1(1-f_2)f_3(\theta_1 - \theta_3)$

element (1, 4): flux = $f_1(1-f_2-f_3)f_4(\theta_1 - \theta_4)$ (flux through atmospheric window)

element (2, 3): flux = $f_2 f_3(\theta_2 - \theta_3)$

element (2, 4): flux = $f_2(1-f_3)f_4(\theta_2 - \theta_4)$

element (3, 4): flux = $f_3 f_4(\theta_3 - \theta_4)$

The flux going to outer space corresponds to node 4, with $\theta_4 = 0$ and f_4=1:

$$\text{OLR} = f_1(1-f_2-f_3)\ \theta_1 + f_2(1-f_3)\theta_2 + f_3\theta_3 \qquad \text{(A2.3)}$$

The back-radiation at the surface corresponds to the -signs of the terms with first node 1 in the surface heat flux, highlighted in yellow:

$$qsurf = f_1 f_2(\theta_1 - \theta_2) + f_1(1-f_2)f_3(\theta_1 - \theta_3) + f_1(1-f_2-f_3)f_4(\theta_1 - \theta_4) \qquad (A2.4)$$

The hypothetical back-radiation, which is not a heat flux, becomes with $\theta_4 = 0$ and $f_4 = 1$:

$$\text{“backrad”} = f_1 f_2 \theta_2 + f_1(1-f_2)f_3\theta_3 \qquad (A2.5)$$

Comparison for two layers

We see that OLR for the two-way Schwarzschild formulation (A2.1) is not exactly equal to the OLR for the one-way finite element formulation (A2.2)!
The difference is due to treatment of the viewfactor, in particular the flux through the atmospheric window, for two layers:

$$\text{FEM:} \quad viewfactorF(1,4) = 1-f_2-f_3 \qquad (A2.6)$$

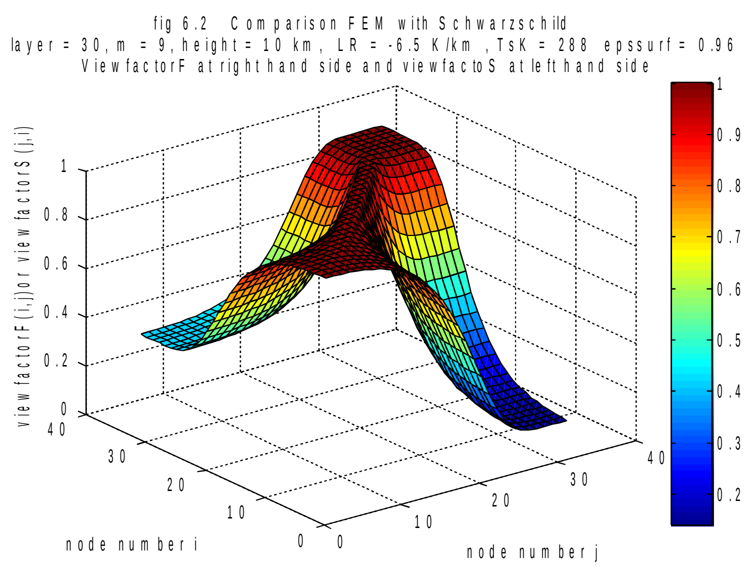
$$\text{Schwarzschild:} \quad viewfactorS(1,4) = (1-f_2)*(1-f_3) = 1- f_2 - f_3 + f_2*f_3 \qquad (A2.7)$$

The difference in back-radiation, that is the difference between (A2.2) and (A2.5), is zero, since in a model with only two layers the view factors between nodes in between do not show up yet.
For models with more layers the difference in results is more pronounce.
Figure A2.2 shows the difference between viewfactorS and viewfactorF for ftot=0.86 and 30 layers.
The viewfactors are defined by:

$$\text{FEM:} \quad viewfactorF(i,j) = 1 - \sum_{k=i+1}^{k=j-1} f_k = 1- f_{i+1} - f_{i+2} \ldots f_{j-1} \qquad (A2.8)$$

$$\text{Schwarzschild:} \quad viewfactorS(i,j) = \prod_{k=i+1}^{k=j-1}(1-f_k) = (1- f_{i+1})(1 - f_{i+2}) \ldots (1-f_{j-1}) \qquad (A2.9)$$

Figure A2.2



viewfactorF on right hand side and viewfactorS on left hand side

We consider the last columns of the matrices viewfactorF and viewfactorS, it are the window factors from nodes i to outer space, windowF(i) and windowS(i) respectively:

$$\text{windowF}(i) = \text{viewfactorF}(i,\text{nods}) = 1 - \sum_{k=i+1}^{k=\text{nods}-1} f_k = 1 - f_{i+1} - f_{i+2} \ldots f_{\text{nods}-1} \quad \text{(A2.10)}$$

$$\text{windowS}(i) = \text{viewfactorS}(i,\text{nods}) = \prod_{k=i+1}^{k=\text{nods}-1} (1-f_k) = (1-f_{i+1})(1-f_{i+2}) \ldots (1-f_{\text{nods}-1}) \quad \text{(A2.11)}$$

In fig A2.3 and A2.4 they are plotted as function of the node number respectively as function of the height z.

Figure A2.3

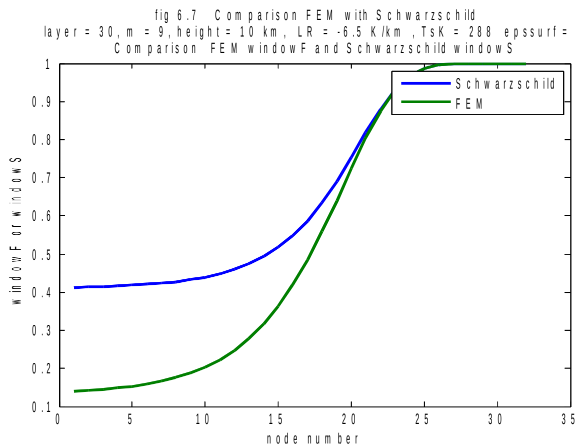
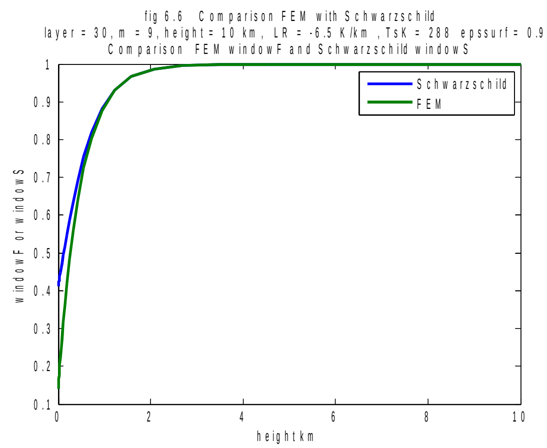
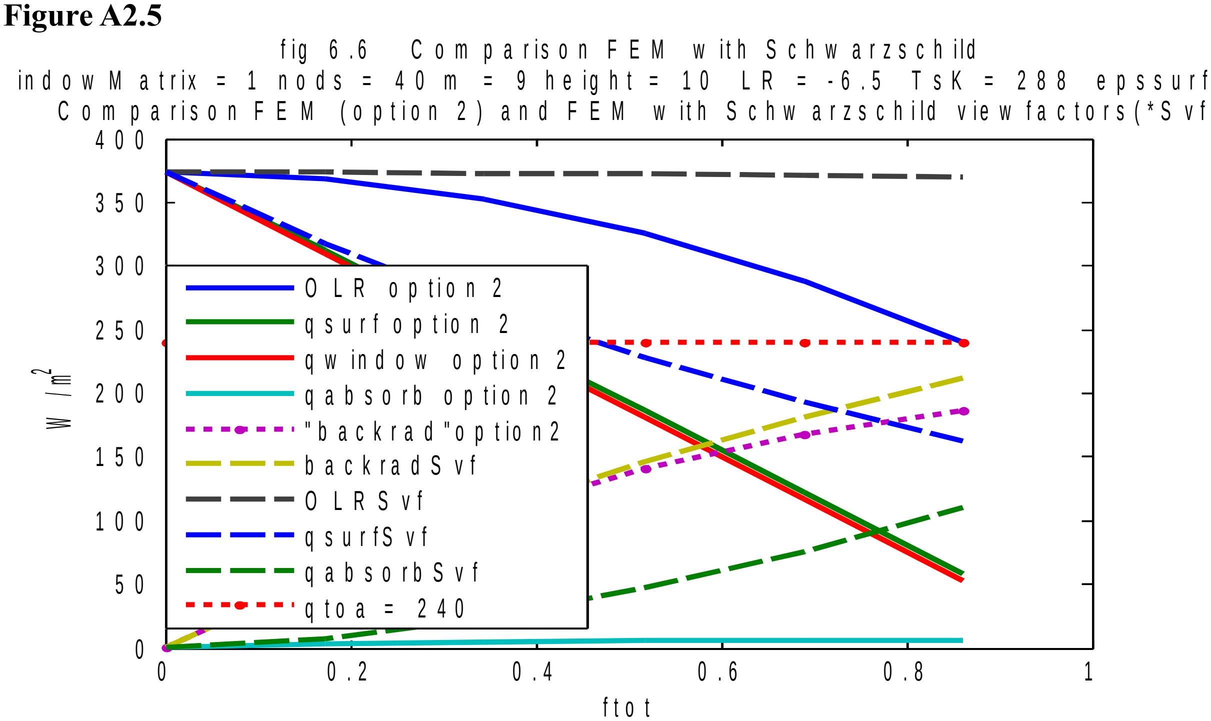


Fig A2.4



The reader might be interested what would become the results if one uses the viewfactorS in the FEM stack model, instead of the viewfactorF.
The result is given in figure A2.5, where the results indicated by option2 are the real FEM results and the results indicated by *Svf those obtained by using the Schwarzschild viewfactorS.

Figure A2.5



We see a disappointing result, in particular for the outgoing long wave radiation: the difference between OLR option 2 and ORLSvf is disturbing. The ORLSvf values seem not to depend on ftot.
And ftot is the optical thickness according to the Schwazschild procedure.
Did nobody remark the fact that OLRvf does not depend on the optical thickness?

The back radiation with the Schwarzschild viewfactors, backradSvf, is closer to the values calculated by the FEM model, backrad option 2, which is not a back-radiation of heat but only a part of the algebraic expression for qsurf option 2.

These differences between the two approaches need more detailed analyses of the Schwarzschild proposal, which we address in appendix 3

Appendix 3

Solution of the Schwarzschild equations at one wavelength interval

In this appendix the proposal of Schwarzschild is studied in detail using matrices for multi-layer models. A more simple discussion for a two layer model was given in appendix 2.

The proposal of Schwarzschild consists of splitting up the radiation in two components: an up-ward component and a down-ward component, U respectively D.
These components are indeed hypothetical. Not everybody agrees that this splitting up was only for reasons of analysis, which back in Schwarzschild's time had to be carried out without computers.
According to the Schwarzschild proposal the absorption by a layer of an absorbing atmosphere can be described by multiplication of the up-going flux U with a factor (1-f), where f is the absorption factor of the layer. The up-ward flux at the surface is supposed to be the Prevost flux: $U_1=\varepsilon\sigma Ts$^4
And in the same way for the downward flux, by assuming it zero at TOA .

We will present in this appendix the original analytical Schwarzschild solution expressed as integrals. The integrals were evaluated by techniques of quadrature available at the time.
We will also use more modern numerical techniques to solve the Schwarzschild equations by an explicit stepping procedure as well as by an implicit scheme based on a variational process of least squares.

Why this zeal on various numerical techniques?
The reason is to try to understand the differences from the results by the FEM formulation of the semi-transparent atmosphere by a stack of grids and the results from the original Schwarzschild procedure which have been programmed by the author, see appendix 5.
The type of results are the same but a difference for the outgoing long wave radiation OLR between the two models is disturbing.

Modification of original Schwarzschild procedure

To bring the results closer to each other, two modifications to the original Schwarzschild procedure are introduced in this paper:

(1) Instead of starting to calculate the up-ward component U by the Prevost component $U_1=\varepsilon\sigma Ts$^4, we make it less hypothetical by subtracting the calculated hypothetical back- radiation at the surface:
the starting condition for the up-ward flux becomes: $U_1 = \varepsilon\sigma Ts$^4 – D_1.

(2) Moreover, we consider the flux through the atmospheric window as a parallel flux not affected by the Schwarzschild procedure of multiplying the flux through an absorbing layer by a factor (1-f).
The starting value for the upward flux becomes: $U_1 = \varepsilon\sigma Ts$^4 – D_1 – qwindow.
Once from this upward flux the outgoing radiation has been evaluated by the Schwarzschild procedure, the parallel flux qwindow has to be added to OLR.

Most readers will agree with the author that these modifications are straight forward!

However, SoD blog [4,5] made already objections to an earlier version of this paper that the splitting up of the radiation in up-ward and down-ward fluxes is called artificial :
it was against what is called the "SoD etiquette"!

We wonder what will now be the objections against the modifications which are making the hypothetical splitting up less hypothetical, since we put the hypothetical components back again together, through the boundary condition!
We see more coherent results when in the boundary conditions for the up-ward flux, the result of the analysis of the down-ward flux is subtracted before carrying out the analysis of the up-ward flux!

Original analytical solution of Schwarzschild

The Schwarzschild equations become with $\beta = N\alpha$ for the up-ward component U respectively the down-ward component D:

$$dU/dz = - \beta (U-B) \qquad (A3.1)$$

$$dD/dz = \beta (D-B) \qquad (A3.2)$$

NB The downward component D is supposed to be positive in the downward direction.

To facilitate the analytical solution for a β depending on z, a coordinate transformation is introduced:

$$d\tau = - \beta dz \quad \text{or} \quad \tau(z) = \int_{z}^{Htoa} \beta\, dz' \qquad (A3.3)$$

Htoa : height of the TOA, top of atmosphere.

Physically the parameter τ is interpreted as optical thickness and represents the amount of absorbers above a point z in the column of air. At TOA we assume τ =0, at the surface $\tau = \tau_{max} = f_{tot}$..

The Schwarzschild equations become:

$$dU/d\tau \;-U = -B \qquad (A3.4)$$

$$dD/d\tau \;+D = B \qquad (A3.5)$$

The solutions become:

$$\exp(-\tau)\,U = -\int_0^{\tau}\exp(-\tau')B\,d\tau' + C1 \qquad (A3.6)$$

$$\exp(\tau)\,D = \int_0^{\tau}\exp(\tau')B\,d\tau' + C2 \qquad (A3.7)$$

It can be inspected by differentiating with respect to τ that it are indeed the solutions of the linear differential equations with variable right hand side B.
The integration constant C2 =0, because at TOA the down-ward flux D=0 and $\tau = 0$.

The integration constant C1 follows from : at z=0 , $\tau = \tau_{max}$.
In the original Schwarzschild procedure the upward flux at the surface is the Prevost flux:

$$U_1 = U(\tau_{max}) = \varepsilon\sigma Ts\text{^}4.$$

$$C1 = \exp(-\tau_{max})\,\varepsilon\sigma Ts\text{^}4 + \int_0^{\tau_{max}}\exp(-\tau')B\,d\tau' \qquad (A3.8)$$

The solutions for U and D become:

$$U(\tau) = \exp(-(\tau_{max}-\tau))\,\varepsilon\sigma Ts\text{^}4 + \exp(\tau)\int_{\tau}^{\tau_{max}}\exp(-\tau')B\,d\tau' \qquad (A3.9)$$

$$D(\tau) = \exp(-\tau)\int_0^{\tau}\exp(\tau')B\,d\tau' \qquad (A3.10)$$

The outgoing long-wave radiation respectively the back-radiation at the surface become:

$$OLR = U(\tau=0) = \exp(-\tau_{max})\varepsilon\sigma Ts\text{^}4 + \int_0^{\tau_{max}}\exp(-\tau')\,B\,d\tau' \qquad (A3.11)$$

$$backrad = D(\tau_{max}) = \int_0^{\tau_{max}}\exp(-(\tau_{max}-\tau'))B\,d\tau' \qquad (A3.12)$$

It is noted that U, D , τ_{max}, τ and the Prevost term $\varepsilon\sigma Ts\text{^}4$, are wavelength dependent and the integrals

have to be evaluated for different wavelengths.
Applying this Schwarzschild solution to the stack model we have to replace B by θ and the optical thickness τ_{max} by ftot(1) .
The integration variables related to optical thickness, τ' and dτ' become:

τ' is replaced by ftot(i):

```
for i =2:nods-2
      ftot(i) = ftot(i-1) – f(i)      (optical thickness at position z(i))
 end
```

dτ' is replaced by f(i)

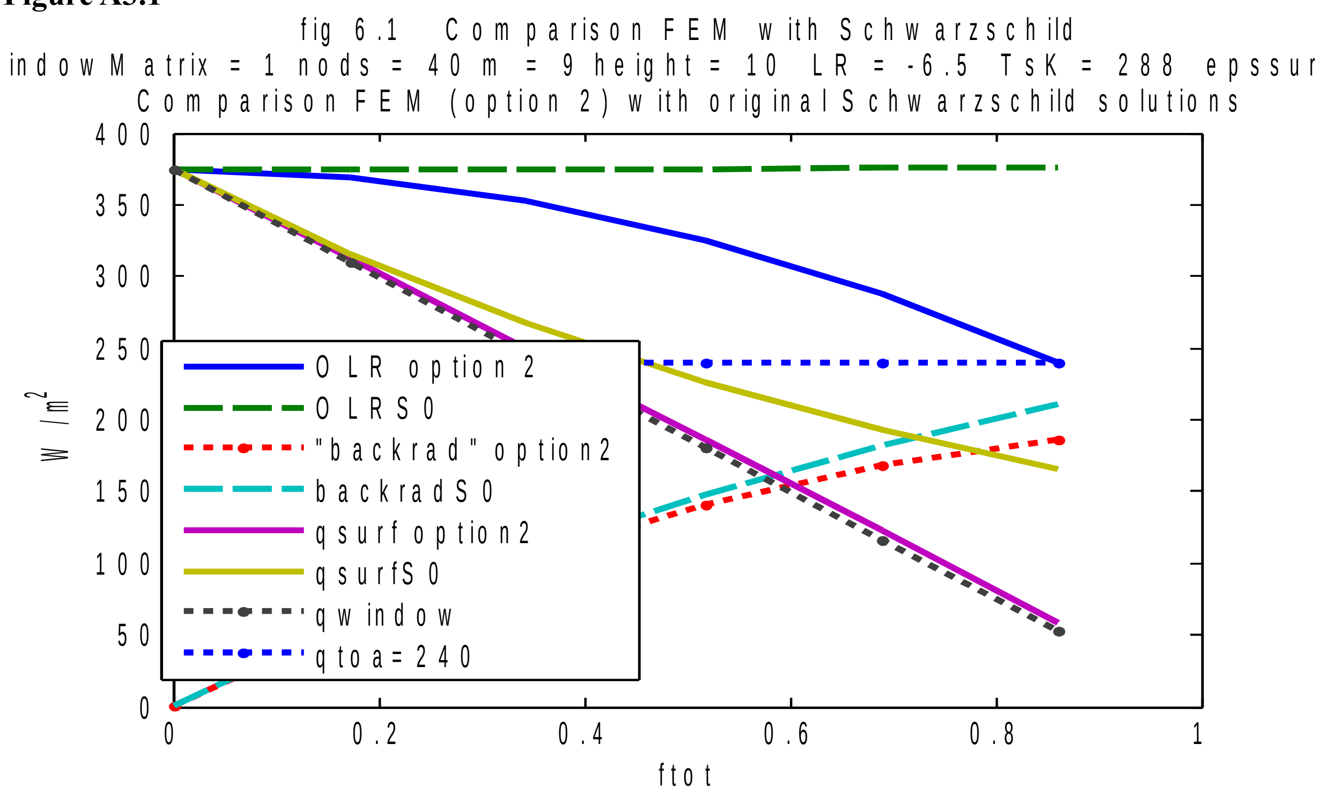
In the equations (A3.11) , (A3.12) the integrals are replaced by discrete summations, denoted by Σ:

$$OLRS0 = (exp(-ftot(1)))\,\varepsilon\sigma Ts\text{^}4 + \sum_{i=2}^{i=nods-1}(f(i)*\theta(i)*exp(-ftot(i))) \quad (A3.13)$$

$$backradS0 = \sum_{i=2}^{i=nods-1}(f(i)*\theta(i)*exp(-(ftot(1)-ftot(i))) \quad (A3.14)$$

These relations were the way how back around 1900 problems were solved, without computers.
The results are given in figA3.1 and compared with the results of the FEM model already given in [1,2] , indicated by option=2, and also depicted in the main text as figure 2.

Figure A3.1



We see that outgoing long wave radiation determined by the original Schwarzschild procedure OLRS0 remains more or less at the level of the Prevost flux at the surface. Not very promising!
OLRS0 is similar to the ORLSvf curve of figure A2.5 of appendix2.
The backradSO is close to the expression defined by the FEM model of the stack, and it is similar to backradSvf of figure A2.5 of appendix 2.

Modified Schwarzschild procedure

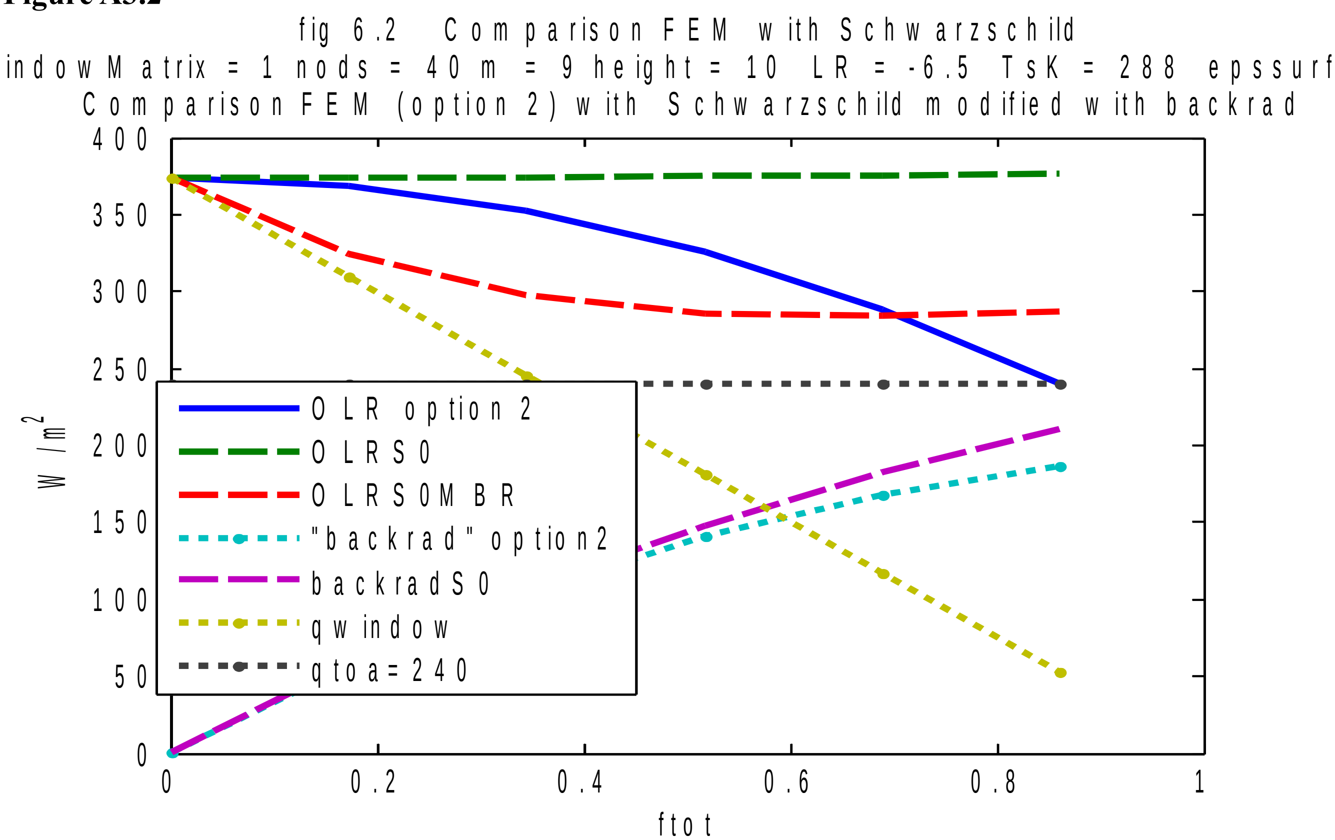
The modified original Schwarzschild solution, due to a modified boundary condition, can be written by subtracting from the Prevost term $\varepsilon\sigma Ts^4$ in (A3.13), the back-radiation term backradS0 and the term qwindow.

We will first apply the backrad modification.
The modified outgoing long wave radiation becomes by modifying for the back-radiation, OLR0MBR

$$OLRS0MBR = (\exp(-ftot(1)))\,(\varepsilon\sigma Ts^4 - backradS0) + \sum_{i=2}^{i=nods-1} (f(i)*\theta(i)*\exp(-ftot(i))) \qquad (A3.15)$$

The results for OLRS0MBR are given in figure A3.2.

Figure A3.2



We see the influence of the modification of subtracting the hypothetical backrad from the hypothetical Prevost flux at the surface.
Still not promising, the red curve OLRS0MBR is not very similar to the blue OLR option 2.

We now apply the second modification of the original Schwarzschild procedure, by considering the flux through the atmospheric window as a parallel radiation, not affected by the IR-active molecules.
We have to subtract also qwindow from the Prevost term.
But once the outgoing long wave radiation has been defined by the Schwarzschild procedure , we have to add qwindow:

$$\text{OLRS0M} = (\exp(-\text{ftot}(1)))\,(\varepsilon\sigma \text{Ts}^4 - \text{backradS0} - \text{qwindow}) + \text{qwindow}$$

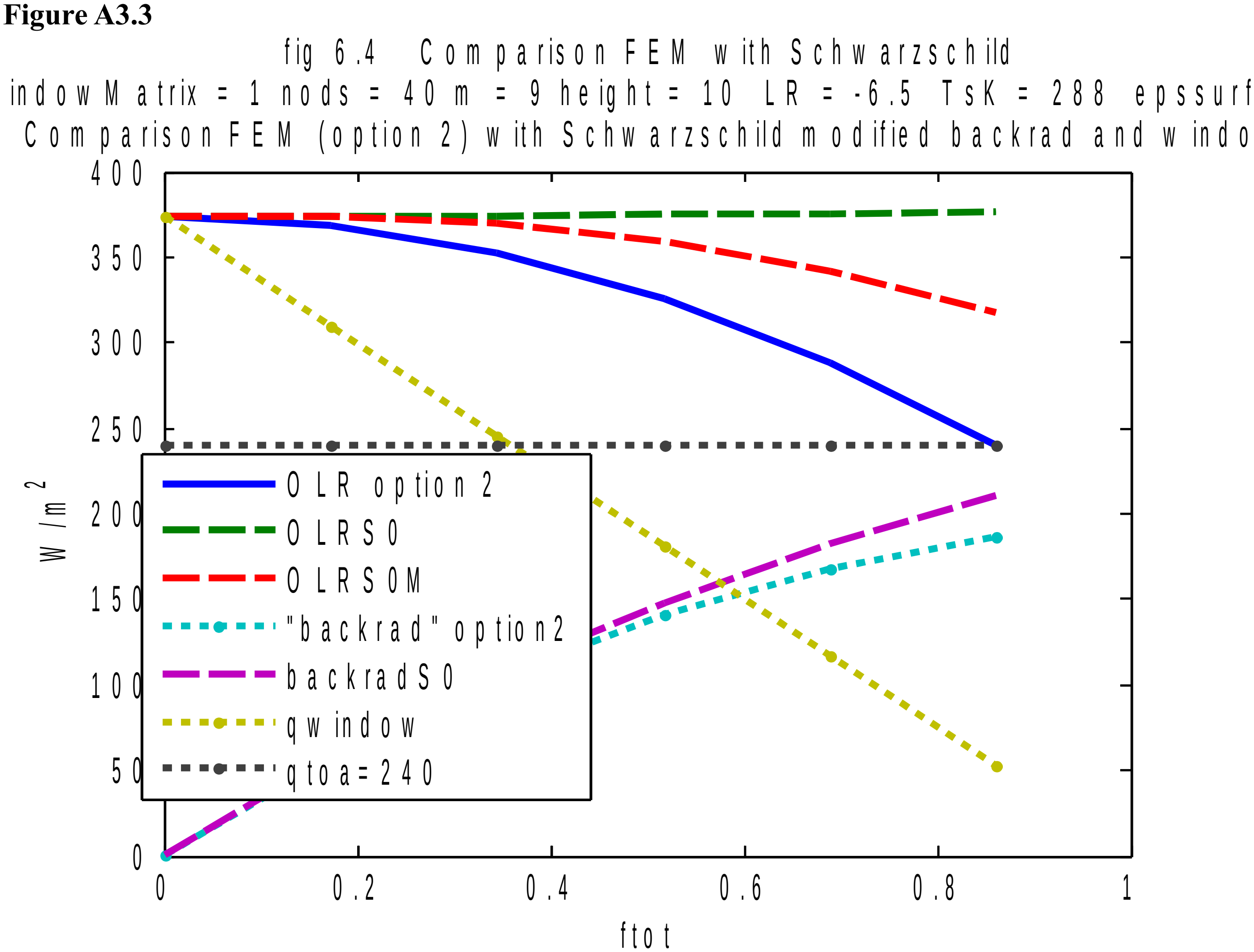
$$\sum_{i=2}^{i=\text{nods}-1} (f(i)*\theta(i)*\exp(-\text{ftot}(i))) \qquad \text{(A3.16)}$$

The parallel flux through the atmospheric window is defined by:

$$\text{qwindow} = (1-\text{ftot})\,\varepsilon\sigma \text{Ts}^4 \qquad \text{(A3.17)}$$

The results of both modifications of the original Schwarzschild procedure, OLRS0M, are given in figure A3.3

Figure A3.3



We observe that with the two modifications the OLRS0M, the outgoing long-wave radiation according to modified Schwarzschild, starts to look like the OLR option 2 from the FEM stack model.

Before the final discussion of the Schwarzschild procedure , we want to show the results of alternative numerical techniques to solve the differential equations. Just to verify that the differences are not due to numerical processes, but to an unfortunate physical Schwarzschild proposal.

Forward stepping numerical procedure, explicit technique

For the discretized stack of the atmosphere with absorption/emission coefficients $f_i = (\beta\,\Delta z)_i$ and Planck function $B_i = \theta_i$, a forward stepping procedure is described to solve equations A3.1 and A3,2, starting from the original respectively the modified boundary conditions for the up-ward and down-ward fluxes:

$$U_{i+1} = U_i\,(1 - f_{i+1}) + f_{i+1}\,\theta_{i+1} \qquad \text{(A3.18)}$$

$$D_i = D_{i+1}(1- f_i) + f_i\,\theta_i \qquad \text{(A3.19)}$$

The original Schwarzschild boundary conditions respectively the modified ones are :

$$U_1 = \varepsilon\,\sigma\,\mathrm{TsK}\text{^}4 \quad \text{or the modified BC} \quad U_1 = \varepsilon\,\sigma\,\mathrm{TsK}\text{^}4 - D_1 - \mathrm{qwindow} \qquad \text{(A3.20)}$$

$$D_{nods-1} = 0 \qquad \text{(A3.21)}$$

The solutions have to be interpreted as :

$$\mathrm{OLRS1} = U_{nods-1} \quad \text{or for modified BC} \quad \mathrm{OLRS1M} = U_{nods-1} + \mathrm{qwindow} \qquad \text{(A3.22)}$$

$$\mathrm{backradS1} = D_1 \qquad \text{(A3.23)}$$

The results are given in figure A3.4 as OLRS1, OLRS1M and backradS1.

Implicit solution

Instead of an explicit stepping process for (A3.1) and (A3.2), an implicit solution technique can be used. The advantage of an implicit solution is that for the same mesh size the solution is more precise, or alternatively a coarser mesh can be used.

Equations (A3.1) and (A3,2) with the mesh and the absorbers $f_i = (\beta\,\Delta z)_i$ can be written as:

up-ward : $\mathbf{A1{*}U = B1{*}\,\Theta = rhs1}$ solution: $\mathbf{U} = \mathrm{inv}(\mathbf{A1})\mathbf{{*}rhs1}$ (A3.24)

down-ward : $\mathbf{A2{*}D = B2{*}\Theta = rhs2}$ solution: $\mathbf{D} = \mathrm{inv}(\mathbf{A2})\mathbf{{*}rhs2}$ (A3.25)

A1, A2, B1, B2 : matrices of order (nods-1) x (nods-1), depending on the absorbing factors f_i

The element matrices can be written as (see text books on finite elements):

$$\mathbf{A1el} = \begin{vmatrix} 1-f+f\text{^}2/3 & -1+f\text{^}2/6 \\ -1+f\text{^}2/6 & 1+f+f\text{^}2/3 \end{vmatrix} \qquad \mathbf{B1el} = \begin{vmatrix} -f/2+f\text{^}2/3 & -f/2+f\text{^}2/6 \\ f/2+f\text{^}2/6 & f/2+f\text{^}2/3 \end{vmatrix}$$

$$\textbf{A2el} = \begin{vmatrix} 1+f+f^2/3 & -1+f^2/6 \\ & \\ -1+f^2/6 & 1-f+f^2/3 \end{vmatrix} \qquad \textbf{B2el} = \begin{vmatrix} f/2+f^2/3 & f/2+f^2/6 \\ & \\ -f/2+f^2/6 & -f/2+f^2/3 \end{vmatrix}$$

The 2x2 element matrices are assembled in the classical way into block diagonal matrices
A1, A2 , B1, B2.

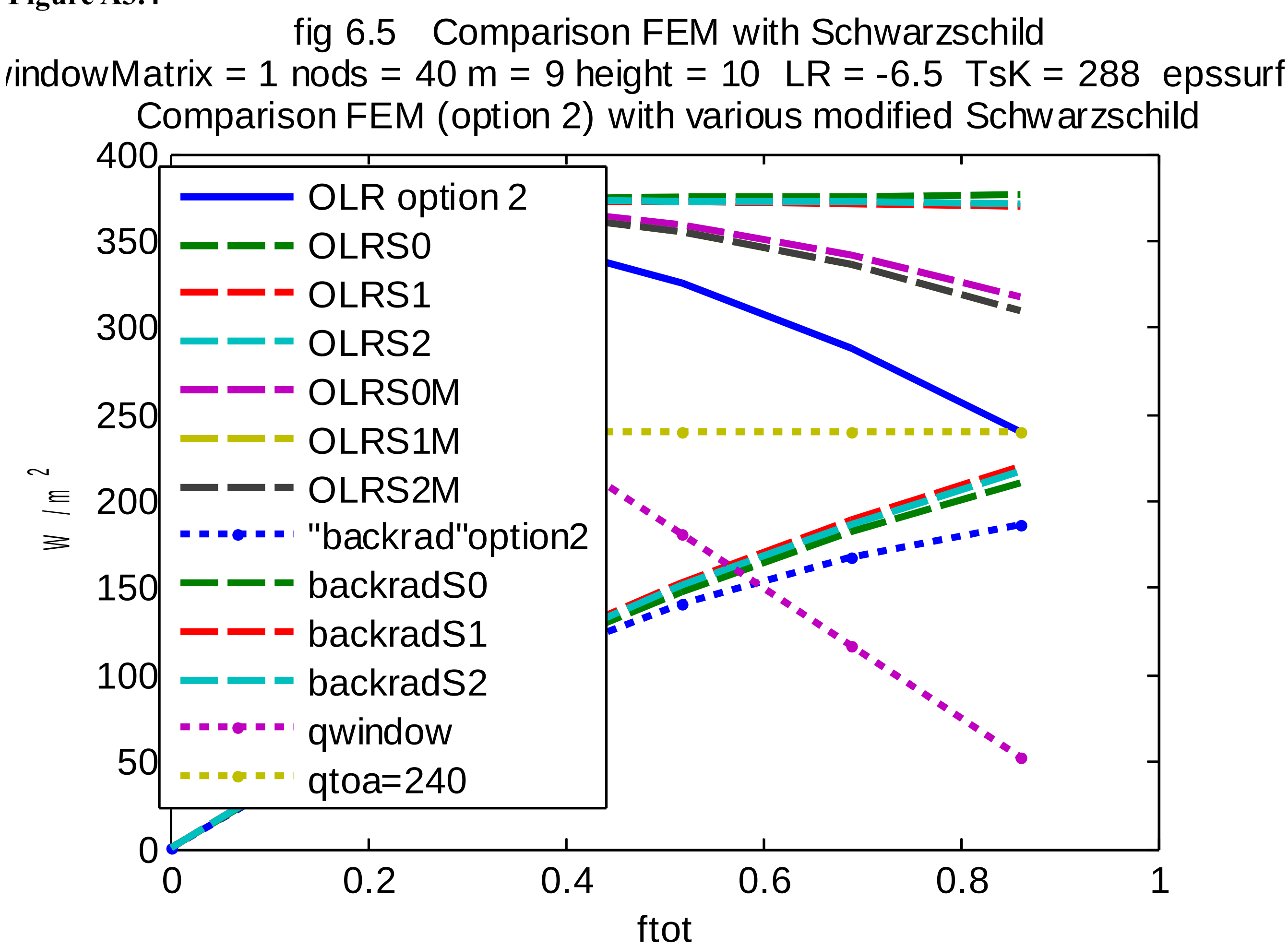
inv(**A1**), inv(**A2)** : inverted matrices taking into account the BC: (A3.20) and (A3.21)

U is the vector of the nods-1 components of the up-ward flux

D is the vector of the nods-1 components of the down-ward flux

Θ is the vector of the nods-1 components of the stack Planck functions
Boundary condition as before, the original Schwarzschild and the modified one:
(A3.18) and (A3.19).

Solutions are interpreted as in the stepping procedure: (A3.22), (A3.23).
In figure A3.4 the results for OLRS2, ORLS2M, backradS2 are included, together with the ones of the original Schwarzschild procedures and the explicit stepping procedure.
Figure A3.4



We see from figure A3.4 that for the original Schwarzschild procedure the 3 results for the outgoing long wave radiation, OLRS0, OLRS1 and OLRS2, are all three about equal to the Prevost surface flux: $\varepsilon \sigma TsK^4$.
All three numerical processes give the same result, the algebra of Schwarzschild is correct the physics is wrong, at least the Schwarzschild physics as presented by SoD in [4,5].

With the two modifications of the boundary condition, as proposed in this paper :

(1) decreasing the starting value of the hypothetical up-ward component with the hypothetical back- radiation D_1 , making the starting value more real

(2) treating the flux through the atmospheric window as a parallel flux, not affected by the IR-active trace gases, making the starting value still more real

the three values OLRS0M, OLRS1M and OLRS2M move towards OLR option 2 of the FEM model, but the measured 240 OLR as given in the K&T papers (see [1]) is not reached by Schwarzschild.

The back-radiation of the Schwarzschild is close to the result for backrad option 2 expression, which is not a back-radiation of heat as already discussed in the main text with equations (14) and (15).

Conclusion concerning the Schwarzschild proposal

The Schwarzschild equation is based on the hypothesis that the component of the hypothetical up-ward Prevost-type of surface flux, augmented by the hypothetical emission from IR-active trace gases, is decreased when passing a layer by a factor (1-f) where f is the absorption factor of the layer.

It turns out that the Schwarzschild procedure as implemented in the SoD blog [4,5] does not give the correct results, at least they are different from the FEM model [2] of which the equations can also be written by means finite differences, as was done in [1].

The FEM model gives results with OLR = 240 W/m^2, for ftot = 0.86 , and surface temperature TsK=288.
The model has been validated by the experimental results of the K&T diagram publications, see [1].

Doubling the CO_2 concentration from 380 ppm to 760 ppm gives a surface temperature sensitivity of 0.03 K. [2]

All the figures presented in this appendix are produced by option 6 of a MATLAB program of which the listing is given in appendix 5.

Appendix 4

View factors are still an issue

In [1] the equations of the stack model were derived by means of finite differences.
The components of the viewfactorF matrix as discussed in appendix 2 were all equal to one

except the windowF vector, being the last column of the matrix viewfactorF, as given in equation (A2.10).
The results were validated by K&T publications see [1], where OLR = 240.
The expression for OLR in the stack model is given by:

$$\mathrm{OLR} = \sum_{i=1}^{i=nods-1} (f(i)*\mathrm{viewfactorF}(i,nods)*f(nods)*(\theta(i) - \theta(nods))) \qquad \text{(A4.1)}$$

For f (nods) = 1 , θ (nods) = 0 and viewfactorF(i,nods) = windowF(i):

$$\mathrm{OLR} = \sum_{i=1}^{i=nods-1} (f(i)*\mathrm{windowF}(i)*\theta(i)) \qquad \text{(A4.2)}$$

We see that indeed for the OLR only the last column of the viewfactorF matrix is used.
The fact that only windowF is used has no effect on OLR.
The qsurf is defined by the equation (14) :

$$\mathrm{qsurf} = \sum_{j=2}^{j=nods} f(1)*\mathrm{viewfactorF}(1, j)*f(j)*(\theta(1) - \theta(j)) \qquad \text{(14) or (A4.3)}$$

The hypothetical back-radiation is defined by the sum of the highlighted terms:

$$\mathrm{backrad} = \sum_{j=2}^{j=nods} f(1)*\mathrm{viewfactorF}(1, j)*f(j)*\ \theta(j) \qquad \text{(15) or (A4.4)}$$

We see in this algebraic viewfactorF components outside the last column, and they were taken as 1 in the finite difference paper [1]. (windowMatrix=0)
In figure A4.1 the results are given with the viewfactorF only different from 1 in the last “nods” column, consisting of windowF(i) as given in equation A2.10. It is the same as already given in [1].

Figure A4.2 gives the results with the complete viewfactorF (windowMatrix=1):

- OLR : the same because it only depends on the last column of viewfactorF = windowF
- qsurf lower — involves now viewfactor components which are nearly all one
- qabsorp lower — idem
- backrad lower — idem
-

Figure A4.1 windowMatrix=0

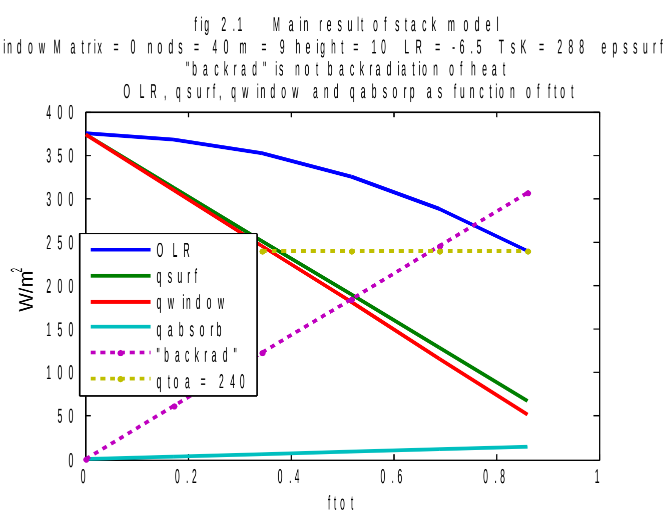
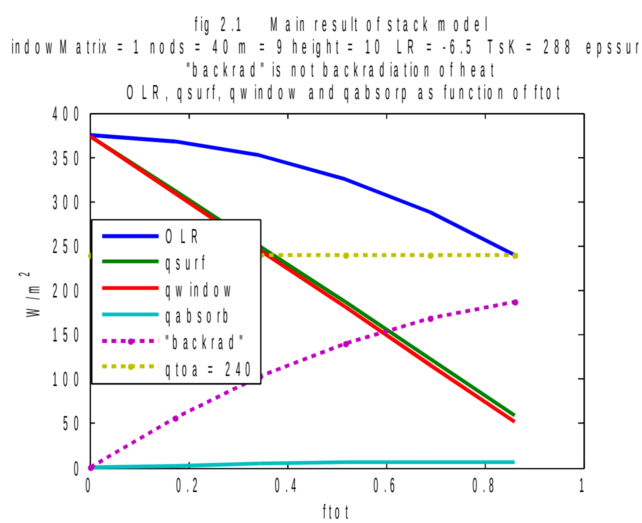


Figure A4.2 windowMatrix=1



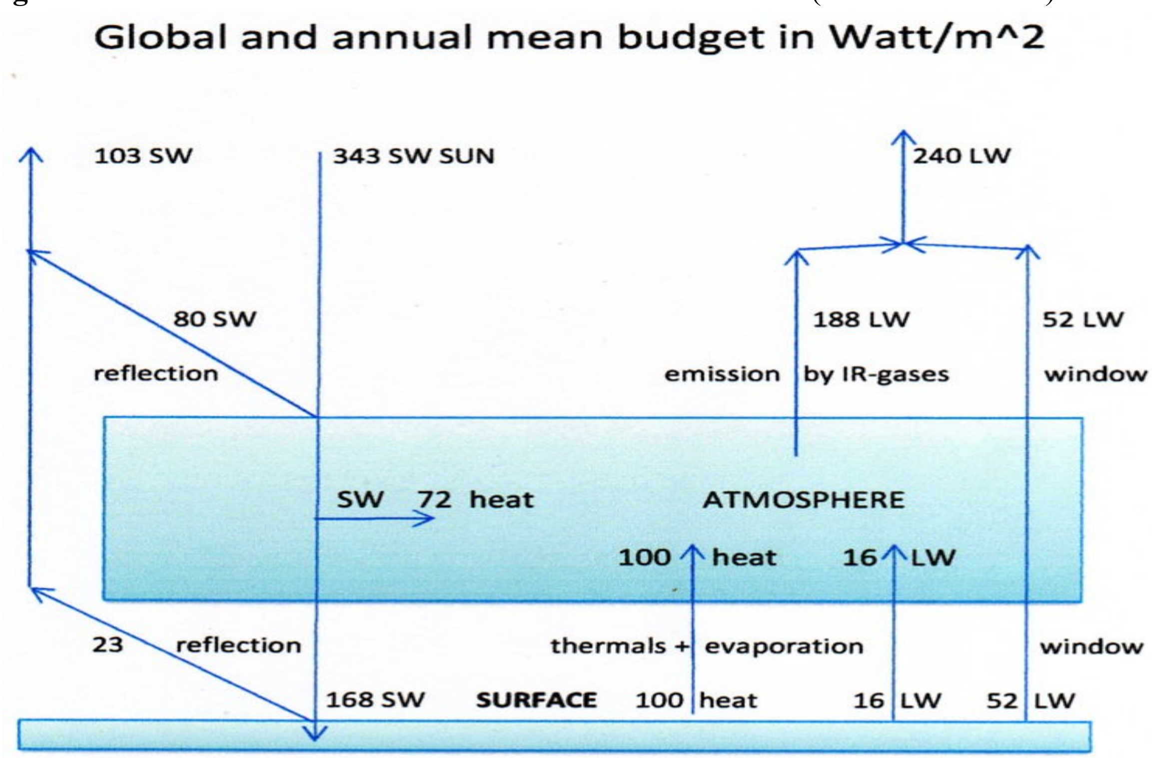
We see here the numbers from ref [1] and we repeat the comparison with K&T as **Table A2-3,** copied here from appendix 2 in ref [1].
The hypothetical back-radiation as defined by the highlighted terms in the algebraic expression (15) or (A4.2), is equal to 306 W/m^2. And with the total viewfactorF matrix (windowMatrix=1) it is about 200 according to figure AA.2.

Table A2-3 heat fluxes Watt/m^2 from reference [1]

source	qtoa OLRtot	qsurf ORLq	mechanims other than LW radiation	absorption LW in atm qatmLW	back-radiation
KT	240	390-324 = 66	169	350	324
VD	235	395-330 = 65	175	355	330
FM	252	378-315 = 63	189	315	315
fig-5-41	240	68	172	23	(306)

It follows clearly from this comparison that the 3 K&T type publications in [1] in fact have used a kind of stack model: from measured temperature distributions heat fluxes are defined, as well as the hypothetical back-radiation.
Unfortunately the term back-radiation is used, and a physical interpretation is given to it.
There is no physical interpretation back-radiation is just an algebraical expression with a negative sign in the surface flux, which is not of the Prevost type.
Figure A4.1 is the basis of the global and annual heat budget as given in figure A4.3 copied from [1].
The figure has to be compared with figure 4 in the main text of the present paper, which is based on a complete viewfactorF matrix. (windowMatix=1).

Figure A4.3 for reduced viewfactorF matrix = windowF vector (windowMatrix=0).



The use of the two different viewfactorF matrices, has little influence on the outgoing OLR in case of a prescribed atmospheric temperature defined by ELR.

It has more effect on the efficiency of heat transport within the atmosphere.
We see that for example the hypothetical back-radiation is 50% higher when the major part of the viewfactorF matrix is kept equal to 1 and only the last column of the matrix has the values of the windowF vector.

We repeat also the hypothetical case of an atmosphere where the IR-active molecules are supposed to be isolated from the 99% bulk of the atmosphere consisting of oxygen and nitrogen.
No heat transport by convection. All heat has to leave the surface by LW radiation.
We get results of the type of figure 1 in the main text which we repeat here in in figure A4.5.
In figure A4.4 we give the same case for components of the viewfactorF matrix closer to one, the components not equal to one are in the last column, represented by the vector windowF.

figure A4.4 window F

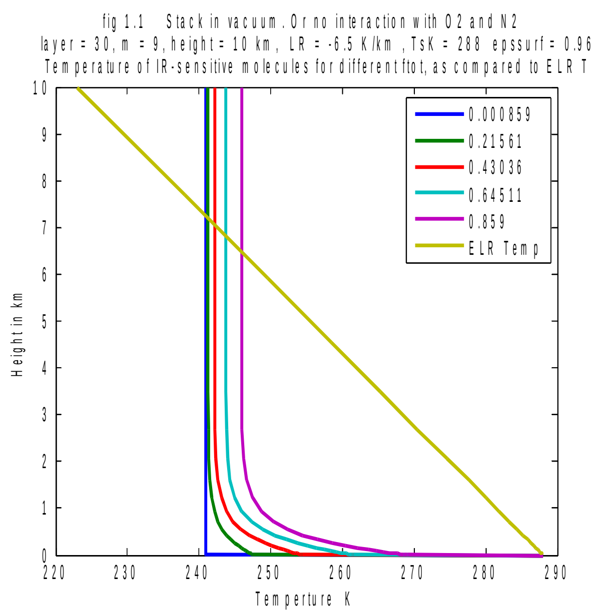
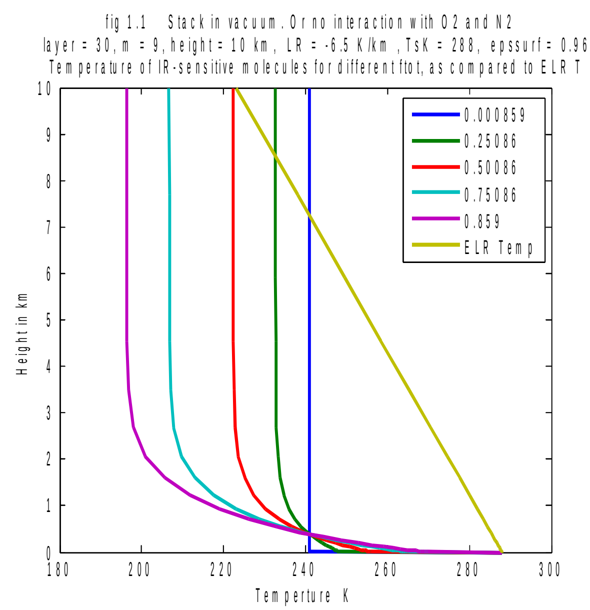


figure A4.5 viewfactorF



We observe a different temperature distribution!
The blue lines correspond to a nearly zero concentration of IR-active trace gases.
They are the same in the two figures.
In figure A4.4 the heat transfer by LW radiation in vertical direction is stronger than in figure A4.5.
In figure A4.4 with a higher heat transfer, for a higher concentration up to ftot = 0.859 the IR-active trace gases are warmer *i.e.* closer to the surface temperature.
In figure A4.5 with a lower heat transfer, for a higher concentration up to ftot=0.859 the IR-active trace gases are colder *i.e* further away of the surface temperature.

However in both cases the IR-active trace gases are cooler than the atmosphere given by the ELR line.
The IR-active gases are not heating up the bulk of 99% of the atmosphere.
It is the other way around, the bulk of the 99% of the atmosphere keeps the IR-active trace gases warm.

The author has included this appendix in order to show that the radiation through the atmosphere is not

yet completely clear: the viewfactorF has an important effect for cases that LW radiation is predominant. In practical cases the LW radiation is not predominant.
The K&T publications with the claimed 320 W/m^2 back-radiation as given in the first paper [1] of the author on the subject, seem to have worked only with the windowF vector, since the backrad of 306 W/m^2 and the OLR = 240 correspond to values found from the stack model with windowMatix=0..

Once the author had developed a numerical tool with FEM , numerical comparisons were easier to carry out. In a first time the author had the idea that using viewfactorF matrix instead of the windowF vector had little influence, which was indeed true for OLR.
But the differences as shown in figures A4.4 and A4.5 need to be studied in more detail.

For the sensitivity analysis, it is obvious that the results are the same for the two view factor distributions: sensitivity for doubling CO2 is 0.1 K. The reader can verify it by running option 5 of the MATLAB program of which the listing is presented in appendix 5.
The reason is that the sensitivity analysis depends on the derivative of OLR with ftot, and the outgoing long wave radiation OLR is given by the last column of viewfactorF = windowF

The author does not have yet any preference for the two different viewfactorF matrices.
Readers who have experimental experience are invited to give suggestions.

Appendix 5

Matlab program to analyze absorption in a semi-transparent atmosphere

Since September 2014 the listing is published in a separate paper:

http://www.tech-know-group.com/papers/Reynen-MATLAB-listing.pdf